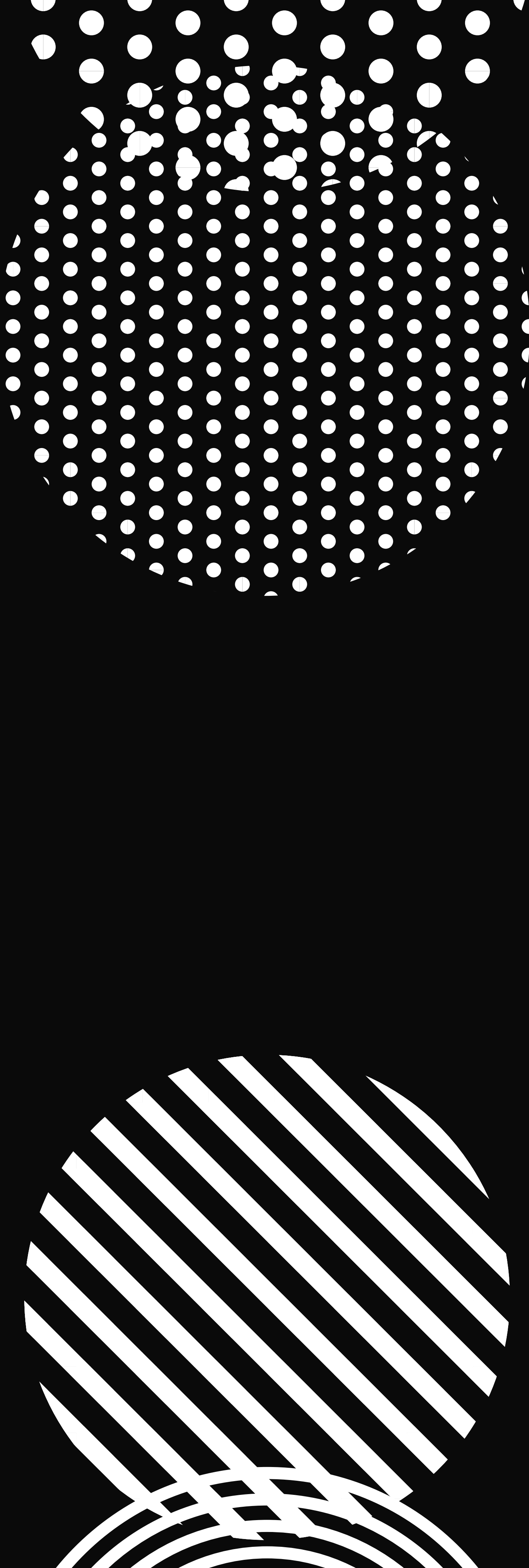


§1 Partitions and equivalence relations

Recall: A partition of a set S is a decomposition

$$S = \bigcup_i A_i$$

w/ $\{A_i\}$ pairwise disjoint (i.e., $A_i \cap A_j = \emptyset$ $i \neq j$).

We write

$$S = \bigsqcup_i A_i$$

Claim from last time: Partitions "=" equivalence relations

:= (Reflexive + symmetric + transitive relations.) $\sim$

e.g.) $S = \{\text{animals}\}$

Equiv. relation

$\left(a \sim b \iff \begin{matrix} a \text{ and } b \\ \text{same species} \end{matrix} \right)$ " $\Longrightarrow$ " $S = \bigsqcup_{\text{Species } X} \left\{ \begin{matrix} \text{Animals} \\ \text{of species} \\ X \end{matrix} \right\}$

To generalize this we give some notation

__Notation:__ Let $\sim$ be an equiv. rel. on S

- for $x \in S$, its __equivalence class__ is

$$[x] = \{ y \in S : x \sim y \}$$

- $S/\sim \; = \{ [x] \}_{x \in S}$ is its set of equivalence classes

__⚠!:__ $S/\sim$ is a __set__ and so does __not__ count repeats. So, if

$S = \{\text{animals}\}$, $\sim$ = "same species"

then Sparky $\neq$ Rex but

$$[\text{Sparky}] = \{\text{Dogs}\} = [\text{Rex}]$$

and so $S/\sim$ doesn't distinguish between [Sparky] and [Rex]. In practice how you find $S/\sim$:

<u>Step 1:</u> Pick $x \in S$ and set aside $[x]$

<u>Step 2:</u> Pick $y \in S - [x]$, set aside $[y]$

Step 3: Pick $z \in S - ([x] \cup [y])$, Set aside $[z]$

⋮

When you've exhausted S you'll have set aside all the elements of $S/\sim$ w/o repeats. The set $x, y, z, \ldots$ is a set of representatives

e.g.) Sparky is the best boi so is representative element of $[\text{Sparky}] = \{\text{Dogs}\}$

Thm: Let S be a set.

(1) If $\sim$ is an equivalence relation on S

$$S = \bigsqcup_{[x] \in S/\sim} [x]$$

is a partition on S.

(2) If

$$S = \bigsqcup_{i} A_i$$

then

$$a \sim b \iff a \text{ and } b \text{ belong to a common } A_i$$

is an equivalence relation.

These operations are inverse

$\boxed{\text{TIG}}$ Consider $\mathbb{R}$ w/ the relation

$$a \sim b \iff a = b + n \text{ w/ } n \in \mathbb{Z}$$

Prove $\sim$ is an equiv. relation, find the partition

$$\mathbb{R} = \bigsqcup_{[x] \in \mathbb{R}/\sim}$$

explicitly. What shape is $\mathbb{R}/\sim$

<u>NB</u>: This illustrates one use of equivalence relations

— formalizes gluing.

(1) is exercise

<u>Pf of Thm. (2):</u> To show $\sim$ is reflexive observe as $x \in S = \bigcup_i A_i$ that $x \in A_i$ for some i so $x \sim x$ as x, x both belong to A_i.

To show symmetric if $a \sim b$ then a and b both belong to a common A_i then b and a belong to a common A_i.

TTG What prop. of a partition have

we not used and how is it relevant to transitivity?,]

To show transitivity, assume $a \sim b$ and $b \sim c$. By definition this means that $\exists\, i, j$ s.t. $a, b \in A_i$ and $b, c \in A_j$. As $b \in A_i \cap A_j$ and $A_i \cap A_j$ if $i \neq j$ we see $i = j$. So $a, c \in A_i$

So $a \sim c$. $\blacksquare$

§ 2 Functions

We now come to one of the raison d'etres of our course: functions.

<u>Def'n</u>: Let X and Y be sets. A <u>relation from X to Y</u> is a subset $R \subseteq X \times Y$.

e.g. If R is a relation on a set, R is a relation from S to S

e.g. If $f(x,y)$ is a 2-variable real relation

$$\Gamma_f = \{((x,y), f(x,y)) : (x,y) \in \mathbb{R}^2\} \subseteq \mathbb{R}^2 \times \mathbb{R}$$

is a relation from $\mathbb{R}^2$ to $\mathbb{R}$

Def'n: If R is a relation from X to Y we call X the domain of R and Y the codomain

We would like to think of functions $f: X \to Y$ as certain relations from X to Y via their graphs.

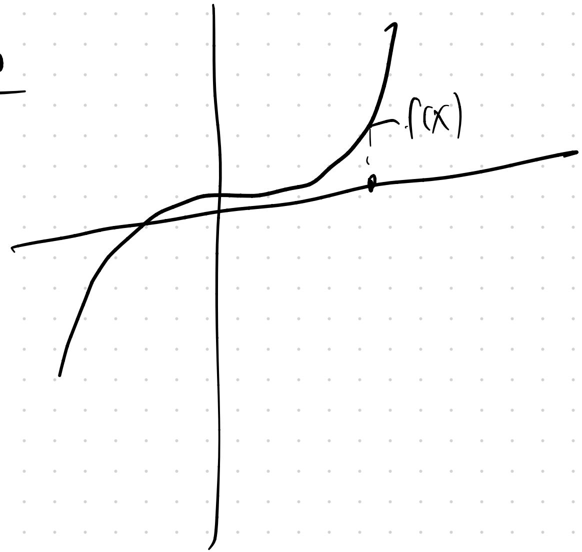
But what properties should this be?

$\boxed{\text{IIG}}$ Are graphs always reflexive? Symmetric? Transitive?

A: None!

So, what properties?

Visually:



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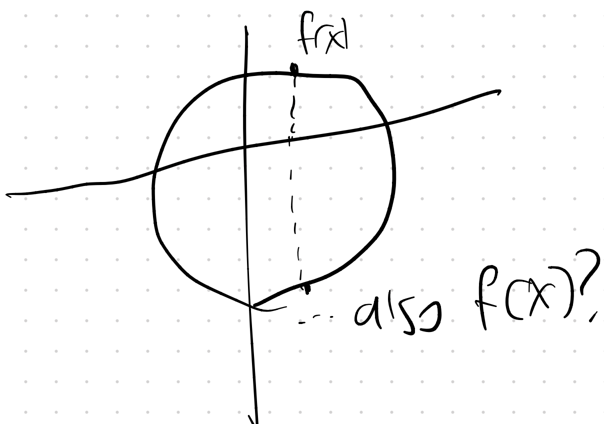
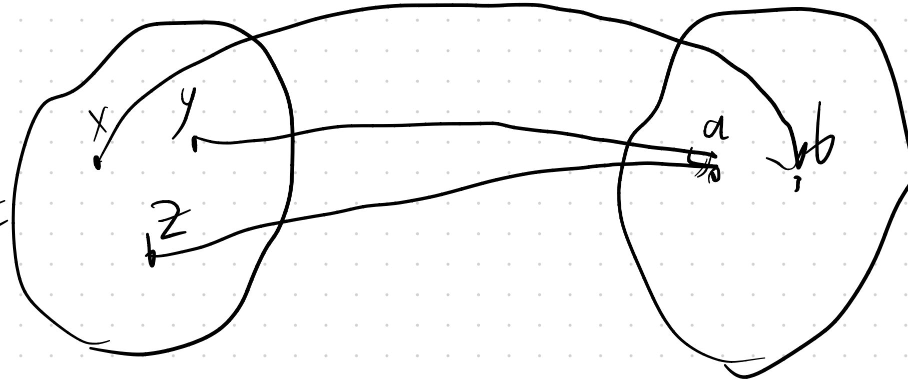


✗

Defn: A function $f: X \to Y$ is a relation R from X to Y s.t. $\forall x \in X$ there is precisely one $y \in Y$ s.t. $(x,y) \in R$. In this case we shorten xRy to $y = f(x)$.

Notation: For a function f we usually denote the relation as Γ_f and call it the graph.

$\boxed{\text{TIG}}$

$f :$ $X =$  $= Y$

What is Γ_f?

<u>Defn:</u> For a function $f : X \to Y$

$S \subseteq X$ and $T \subseteq Y$

- the <u>image</u> of S under X is

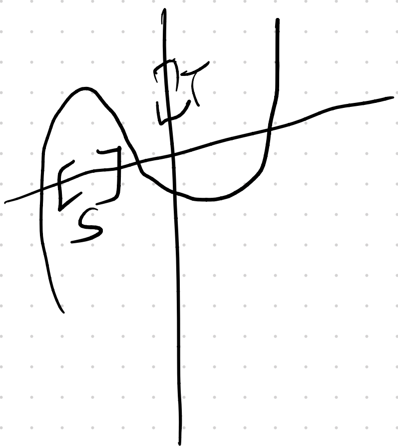
$$f(S) := \{y \in Y : y = f(s) \text{ for } s \in S\}$$

- the <u>preimage</u> of T is

$$f^{-1}(T) := \{x \in X : f(x) \in T\}$$

$\boxed{\text{TIG}}$

- Visually



what is $f(S)$ and $f^{-1}(T)$

__NB:__ As this shows, $f^{-1}(\{x\})$ can be empty / more than one point in general, so is __not__ a function — we will deal later w/

the case when $f^{-1}(\{x\})$ is exactly one point in which case f^{-1} is the <u>inverse</u> <u>function</u>.

- If $f: \mathbb{R}^2 \to \mathbb{R}$ is given by $f(x, y) = x^2 + y^2$, what is $f^{-1}([0,1])$?

<u>Statement</u>: Identities involving Yoga w/ $f(-)$, $f^{-1}(-)$, $\cap$, $\cup$, Δ, ... are

Great test questions.

Prop: Let $f: X \to Y$ be a function. Then, for $S \subseteq X$ and $T \subseteq Y$

$$f(S \cap f^{-1}(T)) = f(S) \cap T$$

Pf: Suppose $y \in$ LHS. Then, $\exists\ x \in S \cap f^{-1}(T)$

s.t. $y = f(x)$. But, as $x \in S$, then $y = f(x) \in f(S)$.
But as $x \in f^{-1}(T)$, also $y = f(x) \in T$. So,
$y \in f(S) \cap T = RHS$.

If $y \in RHS$, then $y \in f(S)$ so there is
some $x \in S$ s.t. $y = f(x)$. But, as $f(x) = y \in T$
we see x is also in $f^{-1}(T)$. So, $x \in S \cap f^{-1}(T)$.
So, $y = f(x) \in f(S \cap f^{-1}(T)) = LHS$ ∎